

# Branching and Iterative Compression

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Seminar on Algorithms, Technion, Winter 18/19

# Outline

- 1 Parameterized Complexity Recap
- 2 Branching Algorithms
  - Vertex Cover
- 3 Iterative Compression
  - 3-Hitting Set

# Parameterized Complexity Basics

## Definition

A **parameterized problem** is a language  $L \subseteq \Sigma^* \times \mathbb{N}$ , when  $\Sigma$  is a fixed, finite alphabet.

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## Definition

We say that an algorithm  $\mathcal{A}$  is a **parameterized algorithm** for  $L \subseteq \Sigma^* \times \mathbb{N}$  if:

- Given  $(x, k) \in \Sigma^* \times \mathbb{N}$  it decide if  $(x, k) \in L$ .
- Its running time is bounded by  $f(k) \cdot |(x, k)|^c$  for a computable  $f$  and a constant  $c$

If such algorithm exists we say that  $L$  is **fixed parameter tractable (FPT)**.

# Vertex Cover

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- We can define

$$VC = \{(G, k) \mid G \text{ has a vertex cover of size at most } k\}$$

- Example (on board)
- NP-hard, No  $(2 - \epsilon)$ -approximation under UGC

# Basic algorithms for Vertex Cover

- Can easily solved in time  $O(m \cdot n^k)$  - **not FPT**.
- Few Observations:
  - 1 If  $E(G) = \emptyset$  and  $k \geq 0$  this is a True instance.

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- We can use these observations to define an algorithm

# Basic algorithms for Vertex Cover

$VC(G, k)$

- 1 If  $E(G) = \emptyset$  and  $k \geq 0$  return **True**
- 2 If  $E(G) \neq \emptyset$  and  $k = 0$  return **False**.
- 3 Select an edge  $(u, v) \in E(G)$   
 return  $VC(G \setminus \{u\}, k - 1)$  or  $VC(G \setminus \{v\}, k - 1)$

Notation:  $G \setminus U = (V(G) \setminus U, \{(u_1, u_2) \in E(G) \mid u_1, u_2 \notin U\})$

Correctness ?

Run time -  $2^k n^{O(1)}$ - FPT

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- 2 **Branching Algorithms**
  - Vertex Cover
- 3 Iterative Compression
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# Branching Algorithms

## The main idea- branching rule

Given  $(I, k) \in \Sigma^* \times \mathbb{N}$ , in polynomial time, either:

- Determine that  $(I, k) \in L$  or  $(I, k) \notin L$ .
- Generate a new set of instances  $(I_1, k_1), \dots, (I_\ell, k_\ell)$  such that

$$(I, k) \in L \iff \exists j, 1 \leq j \leq \ell : (I_j, k_j) \in L$$

The number of instances  $\ell$  may depend on  $(I, k)$ .

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- We can build a *Branching Tree*: for the recursive calls, where  $(I_1, k_1), \dots, (I_\ell, k_\ell)$  are all children of  $(I, k)$
- We would want the instances to be  $(I_1, k_1), \dots, (I_\ell, k_\ell)$  *easier*- often  $0 \leq k_j < k$ .

# A better algorithm for VC

## Corollary

For any  $v \in V(G)$ :

$(G, k) \in VC$  if and only if

$(G \setminus \{v\}, k - 1) \in VC$  **or**  $(G \setminus N(v), k - |N(v)|) \in VC$

## observation

If the maximum degree in  $G$  is 1, then we can decide if  $(G, k) \in VC$  in polynomial time.

# A better algorithm for VC- cont

## **VC2(G,k)**

- ❶ If  $k < 0$  or  $(k = 0 \text{ and } E(G) \neq \emptyset)$  return False
- ❷ If the maximal degree in  $G$  is 1, solve  $(G, k) \in \text{VC}$  and return True/False accordingly.
- ❸ Pick a vertex  $v \in V(G)$  with maximal degree. Return **VC2** $(G \setminus \{v\}, k - 1)$  or **VC2** $(G \setminus N(v), k - |N(v)|)$ .

# A better algorithm for VC- cont

## **VC2**(G,k)

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- ❸ Pick a vertex  $v \in V(G)$  with maximal degree. Return **VC2**( $G \setminus \{v\}, k - 1$ ) or **VC2**( $G \setminus N(v), k - |N(v)|$ ).

- Correctness
- Run time:  $\phi^k n^{O(1)}$ , when  $\phi = \frac{1+\sqrt{5}}{2} \approx 1.61803$

# Even better algorithm for VC

## Observation

If  $G$  has maximal degree 2, we can decide if  $(G, k) \in \text{VC}$  in polynomial time.

## $\text{VC3}(G, k)$

- 1 If  $k < 0$  or  $(k = 0 \text{ and } E(G) \neq \emptyset)$  return False
- 2 If the maximal degree in  $G$  is 2, solve  $(G, k) \in \text{VC}$  and return True/False accordingly.
- 3 Pick a vertex  $v \in V(G)$  with maximal degree. return  $\text{VC3}(G \setminus \{v\}, k - 1)$  or  $\text{VC3}(G \setminus N(v), k - |N(v)|)$ .

# Even better algorithm for VC

## Observation

If  $G$  has maximal degree 2, we can decide if  $(G, k) \in \text{VC}$  in polynomial time.

## $\text{VC3}(G, k)$

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- ❸ Pick a vertex  $v \in V(G)$  with maximal degree. return  $\text{VC3}(G \setminus \{v\}, k - 1)$  or  $\text{VC3}(G \setminus N(v), k - |N(v)|)$ .

- Correctness- same as before
- $T(k) = T(k - 1) + T(k - 3) \rightarrow$  run time  $1.4646^k n^{O(1)}$

# Vertex Cover - Summary

- We got a parameterized algorithm for VC with run time  $1.618^k n^{O(1)}$  (or  $1.4646^k n^{O(1)}$  if the time allowed).
- The best running time is due by Chen, Kanj and Xia. It is a branching algorithm that achieves  $O(1.2738^k + kn)$ .
- There is a theory on solving the running time (Section 3.2 in the book).
- Often **kernelization** is used before the algorithm to reduce the dependency in  $n$  (next week?)

# Summary

- The main idea- split  $(I, k)$  into multiple easier instances  $(I_1, k_1), \dots (I_\ell, k_\ell)$ :

$$(I, k) \in L \iff \exists j : (I_j, k_j) \in L$$

- Better rules can lead to better run times.
- Often need good understanding of the problem to get a branching rule.

# Questions?

Lets take a breaks

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# The 3-Hitting Set problem

## Hypergraph

An hypergraph is a pair  $G = (V, E)$  such that  $V$  is a finite set and  $E \subseteq 2^V \setminus \{\emptyset\}$ .

The rank of  $G$  is  $\max_{e \in E} |e|$ .

An hypergraph of rank 3 or less is 3-hypergraph.

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## Hitting Set

We say that  $S \subseteq V$  is an **hitting set** for an hypergraph  $G = (V, E)$  if for every  $e \in E$  it holds that  $e \cap S \neq \emptyset$ .

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## 3-Hitting Set Problem

Given a 3-hypergraph  $G = (V, E)$  and  $k$ , decide if  $G$  has a hitting set of size at most  $k$ .

$$3HS = \{(G, k) \mid G \text{ has an hitting set of size at most } k\}$$

# The 3-Hitting Set Problem - cont

- A generalization of vertex cover.

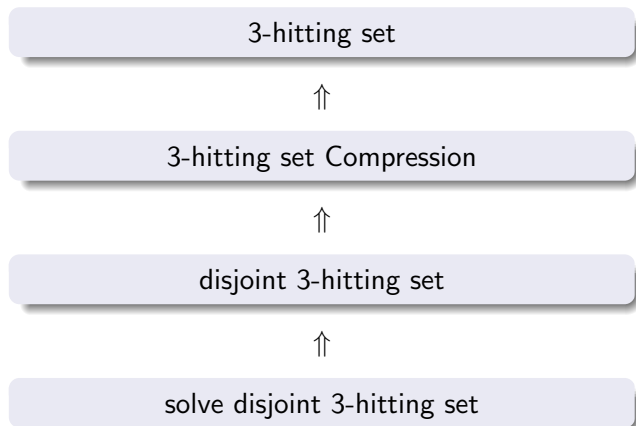
# The 3-Hitting Set Problem - cont

- A generalization of vertex cover.
- Can be solved in time  $3^k n^{O(1)}$  using branching.

# The 3-Hitting Set Problem - cont

- A generalization of vertex cover.
- Can be solved in time  $3^k n^{O(1)}$  using branching.
- We are going to show a  $2.61^k n^{O(1)}$  algorithm using **iterative compression**.

# Iterative Compression- Overview



# Compression

## 3-HS compression problem

Given a 3-hypergraph  $G$ , an hitting set  $X$  of  $G$ ,  $|X| \leq k + 1$ , and  $k$ . Either find a hitting set  $Z$  of  $G$  of size at most  $k$ , or decide no such set exists.

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## Lemma

*If 3-HS compression has an algorithm with run time  $f(k)\text{poly}(n)$  then 3-HS has an algorithm with run time  $f(k)\text{poly}(n)$ .*

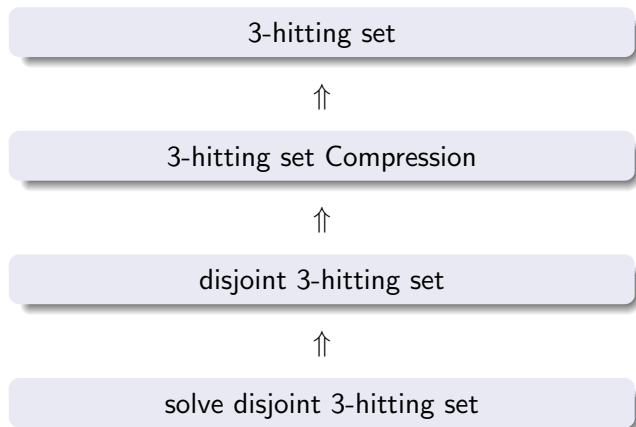
# From 3-HS compression to 3-HS

- Define  $G[U] = (U \cap V(G), \{e \in E(G) | e \subseteq U\})$ .

## Iterative Compression for 3-HS

- 1 Let  $V(G) = \{v_1, v_2, \dots, v_k\}$ ,  $V_i = \{v_1, v_2, \dots, v_i\}$ ,  $G_i = G[V_i]$ .
- 2 Set  $S_0 = \emptyset$  (we refer to this line as iteration 0)
- 3 For  $i$  from 1 to  $n$ :
  - 1 Solve 3-HS compression for  $(G_i, k)$  given the hitting set  $S_{i-1} \cup v_i$ .
  - 2 If got a solution, set  $S_i$  to be the solution. Otherwise return  $(G, k) \notin 3HS$
- 4 return  $S_n$  as a hitting set of size  $k$  for  $G$ .

# Iterative Compression- Overview



# Compression using disjoint

## 3HS- Compression

**Input:** 3-Hypergraph  $G$ , an hitting set  $X, |X| \leq k + 1$

**Output:** An hitting set  $Z, |Z| \leq k$  of  $G$ , or decide  $(G, k) \notin 3\text{HS}$ .

# Compression using disjoint

## 3HS- Compression

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**Output:** An hitting set  $Z, |Z| \leq k$  of  $G$ , or decide  $(G, k) \notin 3HS$ .

## framework

Assume such a set  $Z$  exists.

Guess  $X_Z = X \cap Z$ .

Try to find a an hitting set  $W$  of  $G_X = G \setminus X_Z$ , such that  $W \cap X = \emptyset, |W| \leq k - |X_Z|$ .

Use the fact that  $X \setminus X_Z$  is an hitting set of  $G_X$ .

# Disjoint 3-HS

## Disjoint 3-HS

**input:** A 3-hypergraph  $G$ , an HS  $X$  of  $G$  and  $k$ .

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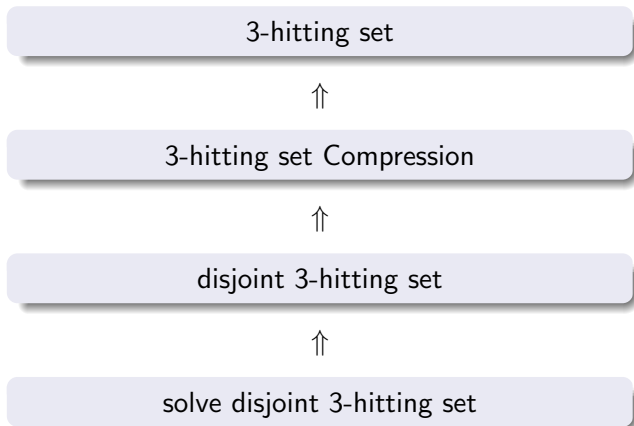
*If Disjoint 3-HS can be solved in time  $\alpha^k \cdot \text{poly}(n)$  then 3-HS compression can be solved in time  $(1 + \alpha)^k \cdot \text{poly}(n)$ .*

# Compression using Disjoint

## 3-HS Compression

- 1 For every  $X_Z \subseteq X$ :
  - 1 Solve disjoint 3-HS for  $G \setminus X_Z$  with  $X \setminus X_Z$  and parameter  $k - |X_Z|$ .
  - 2 If got a solution  $S$ , then return  $S \cup X_Z$
- 2 If failed for every set  $X_Z$ , determine that  $(G, k) \notin 3HS$ .

# Iterative Compression- Overview



# Solving the disjoint problem

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- ① If there is  $e \in E(G)$  such that  $e \subseteq X$  return failure.

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- ② Let  $U = \{v \in V(G) \mid \exists e \in E(G), e \setminus X = \{v\}\}$ . If  $|U| > k$  return failure.

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- ② Let  $U = \{v \in V(G) \mid \exists e \in E(G), e \setminus X = \{v\}\}$ . If  $|U| > k$  return failure.
- ③ Let  $G' = (V', E')$  where  $V' = V(G) \setminus X \setminus U$  and  $E' = \{(u, v) \in V' \mid \exists e \in E(G) : \{u, v\} = e \setminus X\}$ .  
mypause
- ④ Find a vertex cover  $S$  of  $G'$  of size  $k' = k - |U|$ . If one exists, return  $S \cup U$ . Otherwise return failure.

# Iterative compression for 3HS- recap

- We can solve Disjoint 3-HS in time  $1.61^k \cdot \text{poly}(n)$  using a reduction to Vertex Cover.
- We solve 3-HS compression in time  $(1 + 1.61)^k \cdot \text{poly}(n)$ .
- We solve 3-HS in time  $2.61^k \cdot \text{poly}(n)$ .

# Iterative compression-summary

- Two main ideas:
  - Solve a problem by iteratively solving the compression problem.
  - Solve the compression problem using the disjoint problem.
- The disjoint problem is often more simple than the original problem.
- We demonstrated the technique for 3-HS. It can be applied to other problems: feedback vertex set, directed feedback vertex set in tournaments, Odd cycle traversal, etc.

# Questions?

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Thank You!