NP-Completeness

CS 3110

Textbook readings:

Chapter 8
Chapter 9

Overview

- Polynomial-time algorithms
- Optimization problems vs. decision problems
- The class P
- Decision vs. verification
- The class NP
- Polynomial-time reductions
- NP-completeness

NP-complete problems:

- Satisfiability
- Vertex cover
- Hamiltonian cycle
- Subset sum

Polynomial-Time Algorithms

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Problems without such algorithms are considered ***intractable***.

Optimization Problems vs. Decision Problems (1)

Many hard problems are optimization problems:

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Every optimization problem has a corresponding decision problem:

- Given a text and an integer k, decide whether there exists a code that encodes the text in at most k bits.
- Given a graph G and an integer k, decide whether there exists a set S of at most k vertices such that every edge has at least one vertex in S.

Optimization Problems vs. Decision Problems (2)

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Corollary: *If we can provide evidence that a certain decision problem is unlikely to have a polynomial-time algorithm, the corresponding optimization problem is equally unlikely to have a polynomial-time algorithm.*

The Complexity Class P

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Unlikely to be in P:

- Vertex cover
- Satisfiability
- Hamiltonian cycle
- ...

Note: More formally, the classes P and NP are defined as sets of formal languages. Formal languages and decision problems are essentially the same thing. We won't go into this here.

Decision Problems As Sets

It is often convenient to consider a decision problem to be a set $\mathcal{P}$ containing all yes-instances.

Example: For the problem of deciding whether a number is prime, the set $\mathcal{P}$ is the set of all prime numbers $\{2, 3, 5, 7, 11, 13, \ldots\}$.

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Note: If you make this more rigorous, you obtain formal languages and their equivalence to decision problems.

Verification Algorithms

An algorithm $\mathcal{A}$ ***verifies*** a decision problem $\mathcal{P}$ if

$$\mathcal{P} = \{x : \text{there exists a } y \text{ such that } \mathcal{A}(x, y) = \text{“yes”}\}.$$

Think about $\mathcal{A}$ as a proof checker. I claim that $x \in \mathcal{P}$, and I provide a “proof” y. Algorithm $\mathcal{A}$ verifies whether y proves that $x \in \mathcal{P}$. If $\mathcal{A}(x, y) =$ “no”, this does not mean that $x \notin \mathcal{P}$. It only means that y does not prove that $x \in \mathcal{P}$.

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Given a graph G and a set S of vertices of G, it is easy to verify whether any two vertices in S are adjacent in G and whether $|S| \geq k$. An algorithm that does this verifies the class of graphs with independent sets of size at least k.

The Class NP

The complexity class ***NP*** consists of all decision problems $\mathcal{P}$ that satisfy the following condition:

There exists a polynomial-time algorithm $\mathcal{A}$ and a constant c such that

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Lemma: *$P \subseteq NP$.*

Note: Measuring the size of an input is tricky without the notion of formal languages. Just think about it as the number of bits in a reasonable binary encoding of the input.

Polynomial-Time Reductions

An algorithm $\mathcal{A}$ that ***reduces*** a decision problem $\mathcal{P}_1$ to another decision problem $\mathcal{P}_2$ computes a function $\mathcal{A}(x)$, for every x, such that

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Corollary: *If $\mathcal{A}$ is a polynomial-time algorithm and there is no polynomial-time algorithm for $\mathcal{P}_1$, there cannot be a polynomial-time algorithm for $\mathcal{P}_2$.*

NP-Hardness and NP-Completeness

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Corollary: *If we can reduce an NP-hard decision problem $\mathcal{P}$ to another decision problem $\mathcal{P}' \in$ NP using a polynomial-time algorithm, then $\mathcal{P}'$ is NP-complete.*

Satisfiability

The Satisfiability Problem (SAT):

Given a logical formula formed using $\wedge$ (and), $\vee$ (or), and $\neg$ (not); decide whether it is satisfiable, that is, whether there exists a truth assignment to its variables that makes the formula true.

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A formula is said to be in ***3-CNF*** if

$$F = C_1 \wedge C_2 \wedge \cdots \wedge C_k,$$

where

$$C_i = \lambda_{i,1} \vee \lambda_{i,2} \vee \lambda_{i,3}$$

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3-SAT: Given a logical formula in 3-CNF, decide whether it is satisfiable.

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- Membership in NP is easy: given a formula F and a truth assignment to its variables, it takes linear time to verify whether F is satisfied by this assignment.

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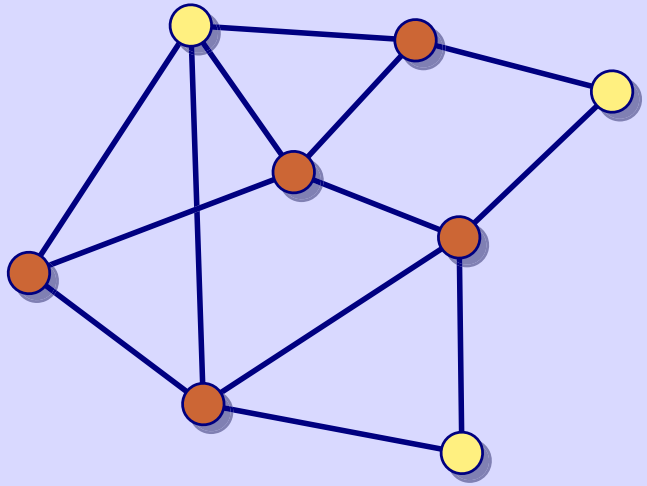
- More precisely, $F_{\mathcal{M}}$ is satisfiable if and only if $\mathcal{M}$ accepts input x.
- The relation between non-deterministic polynomial-time computations and verification, and the construction of $F_{\mathcal{M}}$ are beyond the scope of this course.

Vertex Cover

A ***vertex cover*** of a graph $G = (V, E)$ is a subset S of V such that every edge in E has at least one endpoint in S.



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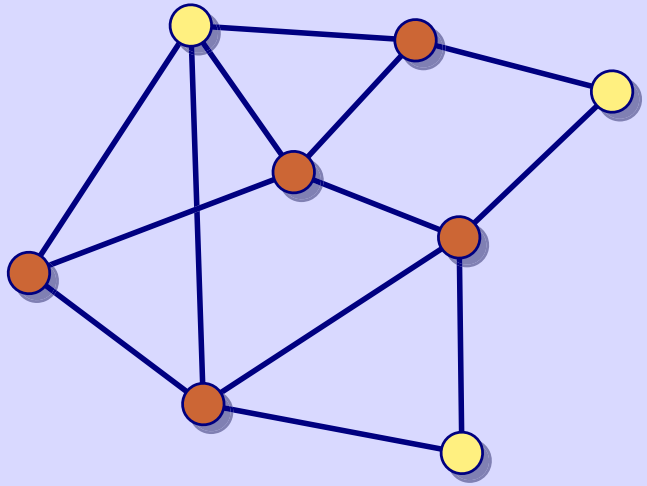

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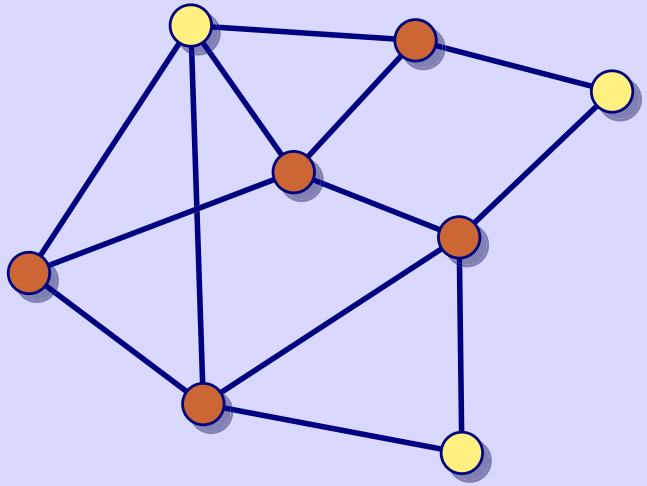

Optimization problem (minimum vertex cover): Given a graph G, determine the size of the smallest vertex cover of G.

Decision problem: Given a graph G and an integer k, determine whether G has a vertex cover of size k.

Vertex Cover is NP-Complete (1)

Reduction from 3-SAT:

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Variable widgets:

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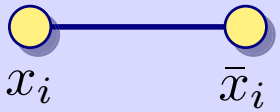
- One edge $(x_i, \bar{x}_i)$



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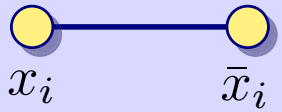
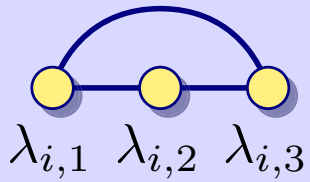


Clause widgets:

- Three literal vertices $\lambda_{i,1}$, $\lambda_{i,2}$, and $\lambda_{i,3}$
- Three edges $(\lambda_{i,1}, \lambda_{i,2})$, $(\lambda_{i,2}, \lambda_{i,3})$, and $(\lambda_{i,3}, \lambda_{i,1})$



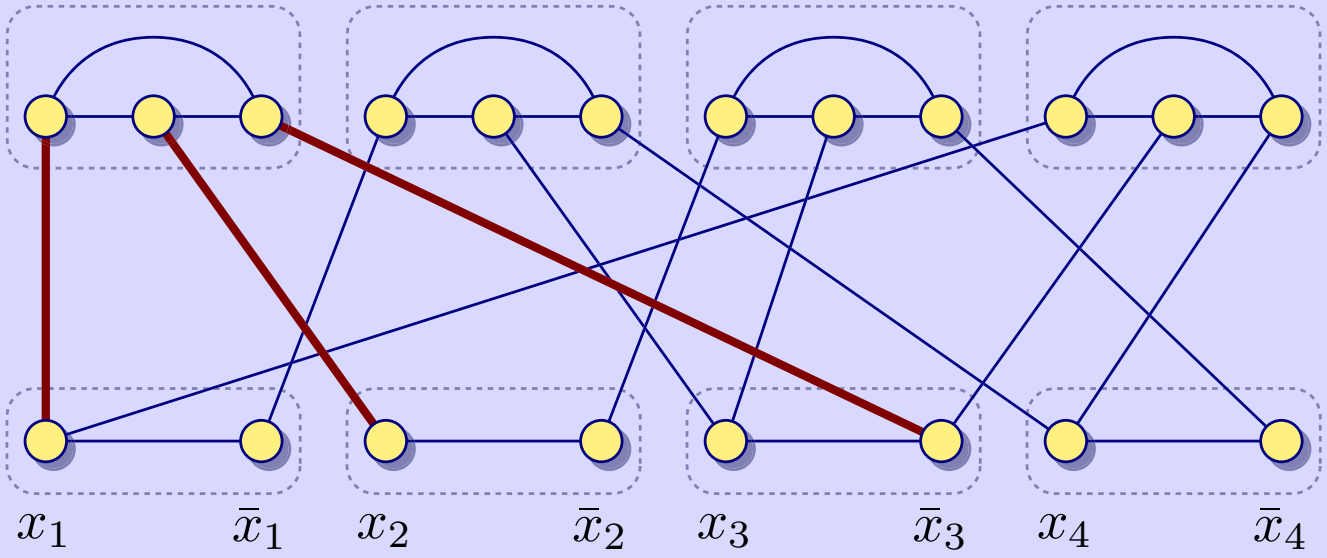
Vertex Cover is NP-Complete (2)

Given: Formula F with n variables and m clauses

$$F = (\boldsymbol{x_1} \vee \boldsymbol{x_2} \vee \boldsymbol{\bar{x}_3}) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

Reduction: Construct a graph G_F with n variable widgets and m clause widgets.

- Connect every literal node $\lambda_{i,j}$ to its corresponding node x_k or $\bar{x}_k$.



Vertex Cover is NP-Complete (3)

Lemma: *Formula F is satisfiable if and only if G_F has a vertex cover of size $n + 2m$.*

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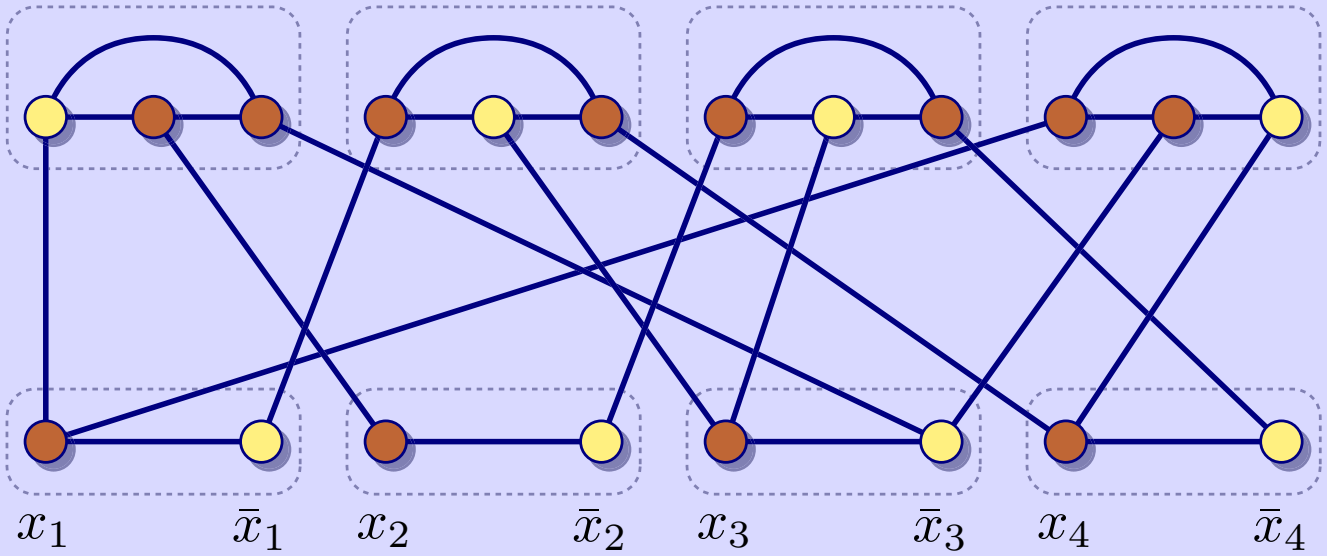
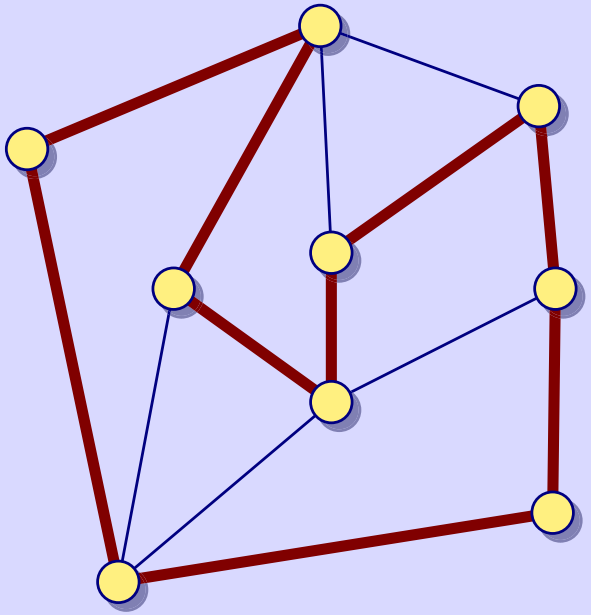


Hamiltonian Cycle

A ***Hamiltonian cycle*** of a graph G is a simple cycle that contains all vertices of G and whose edges are edges of G.

A graph G is ***Hamiltonian*** if it contains a Hamiltonian cycle.



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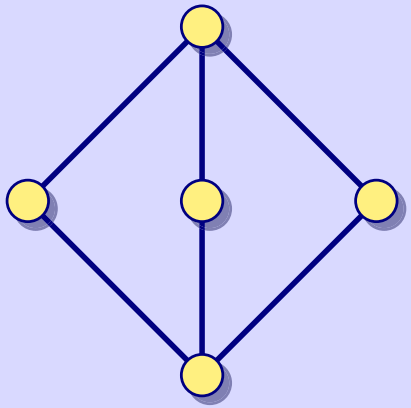

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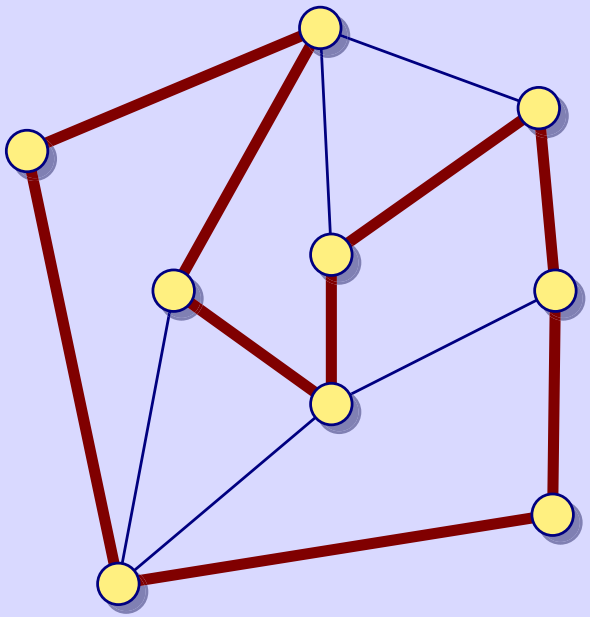

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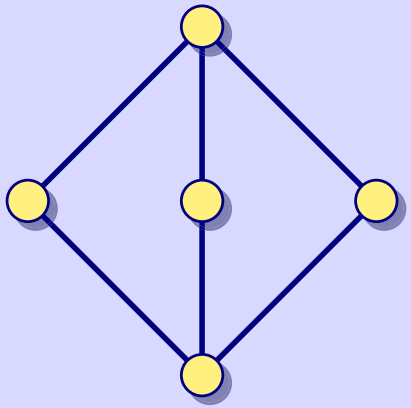

non-Hamiltonian

Decision problem: Given a graph G, decide whether it is Hamiltonian.

Hamiltonian Cycle is NP-Complete (1)

Reduction from vertex cover:

- Build a graph G' from an instance (G, k) such that G has a vertex cover of size k if and only if G' has a Hamiltonian cycle.

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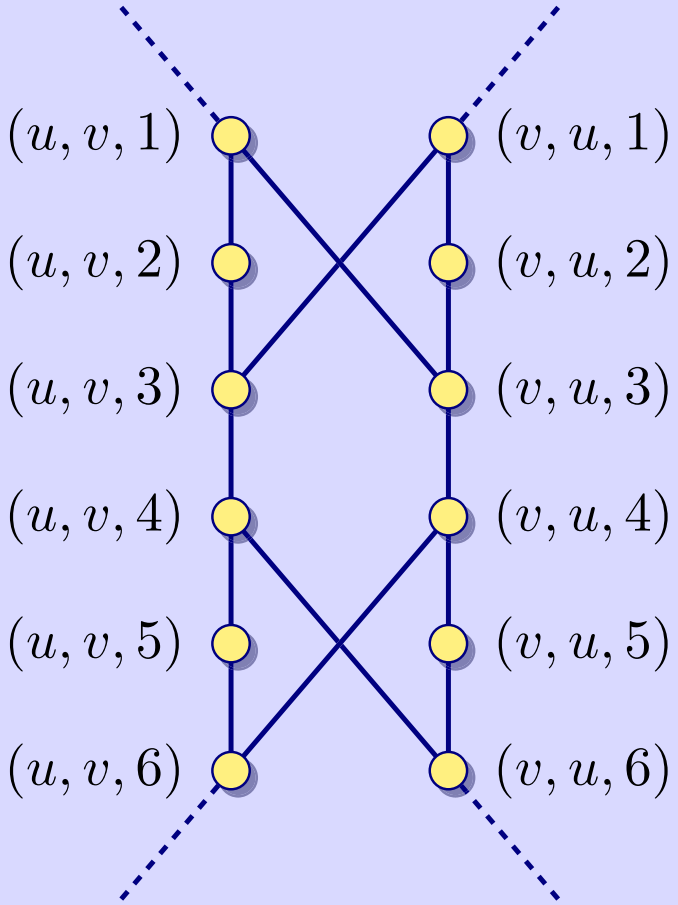
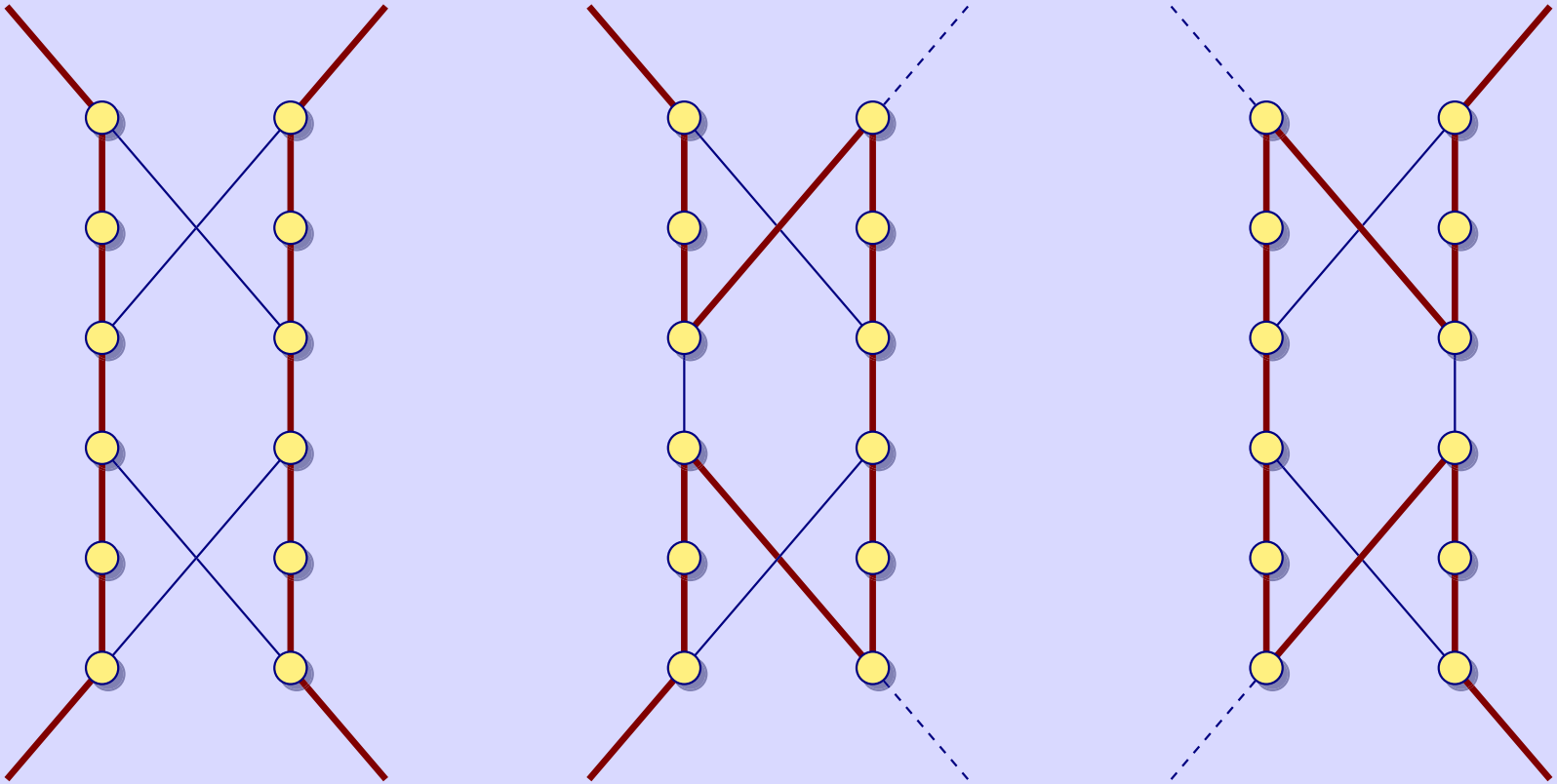
Edge widget for edge (u, v):

- 12 vertices $(u, v, 1), (u, v, 2), \ldots, (u, v, 6)$ and $(v, u, 1), (v, u, 2), \ldots, (v, u, 6)$.
- 14 edges as shown.
- Only $(u, v, 1)$, $(u, v, 6)$, $(v, u, 1)$, and $(v, u, 6)$ have "external" edges (dotted).



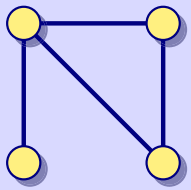
Hamiltonian Cycle is NP-Complete (2)

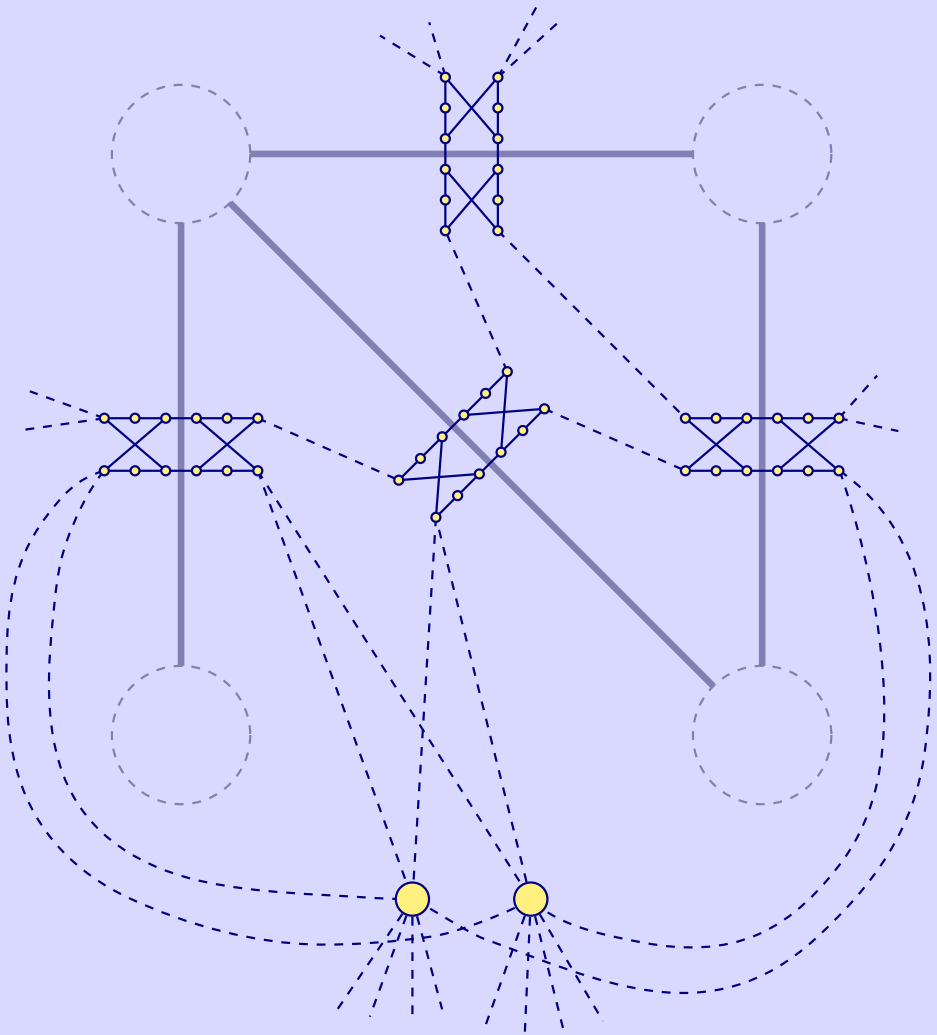
Observation: *A Hamiltonian cycle of a graph built from edge widgets traverses every widget in one of three different ways.*



Hamiltonian Cycle is NP-Complete (3)

Claim: *The graph G' contains a Hamiltonian cycle if and only if G has a vertex cover of size k.*





Hamiltonian Cycle is NP-Complete (3)

Claim: *The graph G' contains a Hamiltonian cycle if and only if G has a vertex cover of size k.*

Subset Sum

Decision problem:

Given a set S of distinct numbers, $S = \{x_1, x_2, \ldots, x_n\}$, and a parameter k, decide whether there exists a subset $S' = \{y_1, y_2, \ldots, y_m\}$ of S such that $\sum_{i=1}^{m} y_i = k$.

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Example:

$$S = \{1, 2, 8, 13\}$$

Set S contains a subset S' whose elements sum to 22, namely $\{1, 8, 13\}$; but there is no subset whose elements sum to 12.

Subset Sum is NP-Complete (1)

Reduction from 3-SAT:

- Given a formula F with n variables and m clauses, we construct a set S_F of $2n + 2m$ numbers with $n + m$ digits in base 10 and a number $t = \sum_{i=0}^{n-1} 10^{i+m} + \sum_{i=0}^{m-1} 4 \cdot 10^i$:

$$t = \underbrace{11\ldots1}_{n \text{ variable digits}} \quad \underbrace{44\ldots4}_{m \text{ clause digits}}$$

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$$t = \underbrace{11\ldots1}_{n \text{ variable digits}} \quad \underbrace{44\ldots4}_{m \text{ clause digits}}$$

- There will be a subset of S_F whose numbers sum to t if and only if F is satisfiable.

Subset Sum is NP-Complete (2)

Literal numbers:

- Two numbers v_i and $\bar{v}_i$ per variable x_i.
- Both have a 1 in the i-th variable digit.
- v_i has a 1 in every digit corresponding to a clause that contains x_i.
- $\bar{v}_i$ has a 1 in every digit corresponding to a clause that contains $\bar{x}_i$.
- All other digits in v_i and $\bar{v}_i$ are 0.

Subset Sum is NP-Complete (2)

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- All other digits in v_i and $\bar{v}_i$ are 0.

Slack numbers:

- Two slack numbers s_i and s'_i per clause.
- s_i has a 1 in the digit corresponding to the clause.
- s'_i has a 2 in the digit corresponding to the clause.
- All other digits in s_i and s'_i are 0.

Subset Sum is NP-Complete (3)

Example:

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

$$v_1 = 00011001$$
$$\bar{v}_1 = 00010100$$
$$v_2 = 00101000$$
$$\bar{v}_2 = 00100010$$
$$v_3 = 01000110$$
$$\bar{v}_3 = 01001001$$
$$v_4 = 10000101$$
$$\bar{v}_4 = 10000010$$

$$s_1 = 00001000$$
$$s'_1 = 00002000$$
$$s_2 = 00000100$$
$$s'_2 = 00000200$$
$$s_3 = 00000010$$
$$s'_3 = 00000020$$
$$s_4 = 00000001$$
$$s'_4 = 00000002$$

Subset Sum is NP-Complete (3)

Example:

$$F = (\boldsymbol{x_1} \vee \boldsymbol{x_2} \vee \bar{x}_3) \wedge (\bar{x}_1 \vee \boldsymbol{x_3} \vee \boldsymbol{x_4}) \wedge (\bar{x}_2 \vee \boldsymbol{x_3} \vee \bar{x}_4) \wedge (\boldsymbol{x_1} \vee \bar{x}_3 \vee \boldsymbol{x_4})$$

- $v_1 = 00011001$
- $\bar{v}_1 = 00010100$
- $v_2 = 00101000$
- $\bar{v}_2 = 00100010$
- $v_3 = 01000110$
- $\bar{v}_3 = 01001001$
- $v_4 = 10000101$
- $\bar{v}_4 = 10000010$

- $s_1 = 00001000$
- $s'_1 = 00002000$
- $s_2 = 00000100$
- $s'_2 = 00000200$
- $s_3 = 00000010$
- $s'_3 = 00000020$
- $s_4 = 00000001$
- $s'_4 = 00000002$

$$v_1 + v_2 + v_3 + v_4 + s'_1 + s'_2 + s_3 + s'_3 + s'_4 = t$$

Subset Sum is NP-Complete (3)

Example:

$$F = (\boldsymbol{x_1} \vee \boldsymbol{x_2} \vee \bar{x}_3) \wedge (\bar{x}_1 \vee \boldsymbol{x_3} \vee \boldsymbol{x_4}) \wedge (\bar{x}_2 \vee \boldsymbol{x_3} \vee \bar{x}_4) \wedge (\boldsymbol{x_1} \vee \bar{x}_3 \vee \boldsymbol{x_4})$$

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- $v_4 = 10000101$
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- $s'_3 = 00000020$
- $s_4 = 00000001$
- $s'_4 = 00000002$

$$v_1 + v_2 + v_3 + v_4 + s'_1 + s'_2 + s_3 + s'_3 + s'_4 = t$$

Lemma: *The Set S_F contains a subset that sums to t if and only if F is satisfiable.*

Summary

Many important problems are NP-complete.

Examples:

- Satisfiability
- Clique
- Vertex cover
- Independent set
- Hamiltonian cycle
- Subset sum
- ...

These problems are unlikely to be solvable in polynomial time.

Alternatives:

- Approximation algorithms (CS 4113)
- Parameterized algorithms